Exercise 64

Evaluate $\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$.

Solution

Plugging in $x = 2$ doesn't work because the denominator becomes zero, using the Quotient Law doesn't work because the limit of the denominator as $x \to 2$ is zero, there doesn't seem to be a way to cancel out the denominator by factoring the numerator, and multiplying the numerator and denominator by 1 doesn't simplify the function. What's next is to look for a substitution.

$$\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = \lim_{x\to 2} \frac{\sqrt{3+(3-x)}-2}{\sqrt{3-x}-1}$$

Let $u = \sqrt{3-x}$. Then $u^2 = 3 - x$. Note that as $x \to 2$, $u \to 1$.

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} &= \lim_{u\to 1} \frac{\sqrt{3+u^2}-2}{u-1} \\
&= \lim_{u\to 1} \frac{\sqrt{3+u^2}-2}{u-1} \times \frac{\sqrt{3+u^2}+2}{\sqrt{3+u^2}+2} \\
&= \lim_{u\to 1} \frac{\left(\sqrt{3+u^2}-2\right)\left(\sqrt{3+u^2}+2\right)}{(u-1)\left(\sqrt{3+u^2}+2\right)} \\
&= \lim_{u\to 1} \frac{(3+u^2)-4}{(u-1)\left(\sqrt{3+u^2}+2\right)} \\
&= \lim_{u\to 1} \frac{u^2-1}{(u-1)\left(\sqrt{3+u^2}+2\right)} \\
&= \lim_{u\to 1} \frac{(u+1)(u-1)}{(u-1)\left(\sqrt{3+u^2}+2\right)} \\
&= \lim_{u\to 1} \frac{u+1}{\sqrt{3+u^2}+2} \\
&= \frac{1+1}{\sqrt{3+1^2}+2} \\
&= \frac{1}{2}
\end{aligned}$$